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A Wolff Theorem for Interpolation Methods Associated to Polygons II

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We give a Wolff Theorem in the quasi-Banach case that improves that known before for the Banach case. We include an application of this result to scales of interpolation spaces. © 2000 Academic Press

INTRODUCTION

In 1982 Thomas Wolff published in [19] a sort of converse of the reiteration theorem for the real and complex method. Since then Wolff theorem has been a matter of study for several authors, see [16, 6, 8, 15, 20].

In [6] Cobos and Peetre extended Wolff theorem to a multidimensional context. More precisely they gave a Wolff theorem for maximal and minimal Aronszjan–Gagliardo functors and specialized this general result to Sparr's methods (see [18]) in the Banach case. A different approach is given to cover the quasi-Banach case. In that paper is suggested that an extension of Wolff theorem to Fernandez's spaces (see [14]) could be made.

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The polygons methods were introduced by Cobos and Peetre in [7]. These methods coincide with Sparr method when the polygon is the simplex (N = 3) and with Fernandez's spaces when the polygon is the unit square (N = 4).

In 1992 Cobos and one of the present authors gave, in the Banach case, a Wolff theorem for the polygons methods, in the case of three or four vertices. The main obstacle to extend the theorem to polygons of an arbitrary number of vertices was the reiteration theorem. The quasi-Banach case for the polygons method remained open, see [8].

In Section 1 we include a reiteration result, Theorem 1.3 (which was independently stated by Ericsson, see [13]), relating methods associated to different polygons. This result allows us to forget about the hypothesis on the parameters. This advantage in the reiteration theorem, not to have to pay attention to the hypothesis on the parameters, allows us to extend Wolff theorem to polygons of any number of vertices and any number of intermediate spaces.

We also extend to the polygons methods recent maximal and minimal descriptions, in the sense of Aronszjan–Gagliardo for the classical real method for couples in the category of quasi-Banach spaces, see [4] and [17]. These descriptions support the arguments given by Cobos and Peetre in [6] to establish Wolff Theorem in the category of quasi-Banach spaces.

In Section 2 we give some applications of Wolff theorem studying scales of interpolation spaces. We adapt the definition of interpolation scale made by Cobos and Peetre in [6] to the polygons methods, and treat two questions made there for scales of interpolation spaces. More precisely, we show that certain types of scales can be pasted together and that we can fill in the gaps generated by this join.

1. THE METHODS ASSOCIATED TO POLYGONS

We start describing the interpolation methods associated to polygons. Subsequently let $\Pi = \overline{P_1, ..., P_N}$ be a convex polygon in the affine plane $\mathbb{R}^2$ with vertices $P_j = (x_j, y_j)$, $j = 1, ..., N$. Let $\bar{A} = \{A_1, ..., A_N\}$ be a Banach N-tuple. That is to say, a family of N Banach spaces all of them continuously embedded in a common linear Hausdorff space. It is useful to think of the space A_j of the N-tuple $\bar{A}$ as sitting on the vertex P_j of Π. Given $t, s > 0$ and $a \in \Sigma(\bar{A})$ we define the K-functional of a as

$$K(t, s, a; \bar{A}) = \inf \left\{ \sum_{j=1}^{N} t^{x_j} s^{y_j} \|a_j\|_{A_j}, a = a_1 + \cdots + a_N, a_j \in A_j \right\}.$$

This provides a family of equivalent norms on the sum $\Sigma(\bar{A})$. Now given $(\alpha, \beta) \in \operatorname{Int} \Pi$ and $1 \leqslant q \leqslant \infty$ the K-interpolation space $\bar{A}_{(\alpha, \beta), q; K}$ is formed by all those elements in $\Sigma(\bar{A})$ for which the norm

$$\|a\|_{(\alpha, \beta), q; K} = \left(\sum_{m, n \in \mathbb{Z}} (2^{-\alpha m - \beta n} K(2^m, 2^n, a; \bar{A}))^q \right)^{1/q}$$

if finite. If $q = \infty$

$$\|a\|_{(\alpha, \beta), \infty; K} = \sup_{m, n \in \mathbb{Z}} \{2^{-\alpha m - \beta n} K(2^m, 2^n; \bar{A})\}.$$

We also have a family of norms on the intersection, $\Delta(\bar{A}) = A_1 \cap \cdots \cap A_N$, defined by means of the J-functional

$$J(t, s, a; \bar{A}) = \max_{1 \leqslant j \leqslant N} \{t^{x_j} s^{y_j} \|a\|_{A_j}\}$$

for $t, s > 0$ and $a \in \Delta(\bar{A})$. Given $(\alpha, \beta) \in \operatorname{Int} \Pi$ and $1 \leqslant q \leqslant \infty$ the J-interpolation space $\bar{A}_{(\alpha, \beta), q; J}$ is formed by all those elements $a \in \Sigma(\bar{A})$ which can be represented as $a = \sum_{m, n \in \mathbb{Z}} u_{m, n}$ (convergence in $\Sigma(\bar{A})$) verifying that

$$\left(\sum (2^{-\alpha m - \beta n} J(2^m, 2^n, u_{m, n}))^q \right)^{1/q} < \infty.$$

The norm being

$$\|a\|_{(\alpha, \beta), q; J} = \inf \left\{ \left(\sum (2^{-\alpha m - \beta n} J(2^m, 2^n, u_{m, n}))^q \right)^{1/q} \right\}$$

where the infimum extends over all representations of a as above. In case $q = \infty$ we replace the sums by *sup* as before.

These definitions can be extended to the category of quasi-Banach linear spaces. We work with N-tuples of quasi-Banach linear spaces. In fact in this case the definition makes sense for any value of the parameter $q > 0$. For these values of the parameter q the functionals $\|\cdot\|_{(\alpha, \beta), q; K}$ and $\|\cdot\|_{(\alpha, \beta), q; J}$ are quasi-norms. Subsequently we will work in the context of quasi-Banach linear spaces.

In contrast with the classical real method, where J- and K-interpolation methods coincide with equivalence of norms, J- and K-spaces do not necessarily coincide for the polygons methods, instead we have the continuous inclusion

$$\bar{A}_{(\alpha, \beta), q; J} \hookrightarrow \bar{A}_{(\alpha, \beta), q; K}$$

for $(\alpha, \beta) \in \operatorname{Int} \Pi$ and $0 < q \leqslant \infty$. When J and K-method coincide on an N-tuple $\bar{A}$, we say it has the J-K equivalence property.

Let $\bar{B} = \{B_1, ..., B_N\}$ be another quasi-Banach N-tuple. By an operator $T: \bar{A} \to \bar{B}$ we mean a bounded linear operator $T: \Sigma(\bar{A}) \to \Sigma(\bar{B})$ whose restriction to A_j defines a bounded linear operator $T: A_j \to B_j$, $1 \leqslant j \leqslant N$. Let $(\alpha, \beta) \in \operatorname{Int} \Pi$ and $0 < q \leqslant \infty$, if $T: \bar{A} \to \bar{B}$ then the interpolated operators

$$T: \bar{A}_{(\alpha, \beta), q; K} \to \bar{B}_{(\alpha, \beta), q; K}$$

$$T: \bar{A}_{(\alpha, \beta), q; J} \to \bar{B}_{(\alpha, \beta), q; J}$$

are bounded. The following estimate, that can be found in [9] for $1 \leqslant q \leqslant \infty$ and Banach tuples, also holds for $0 < q \leqslant \infty$ and quasi-Banach tuples.

$$\|T\|_{\bar{A}_{(\alpha, \beta), q; J}, \bar{B}_{(\alpha, \beta), q; J}} \leqslant \max_{(i, j, k) \in \mathscr{P}_{(\alpha, \beta)}} \{\|T\|^{c_i}_{A_i, B_i} \|T\|^{c_j}_{A_j, B_j} \|T\|^{c_k}_{A_k, B_k}\} \tag{1}$$

$$\|T\|_{\bar{A}_{(\alpha, \beta), q; K}, \bar{B}_{(\alpha, \beta), q; K}} \leqslant \max_{(i, j, k) \in \mathscr{P}_{(\alpha, \beta)}} \{\|T\|^{c_i}_{A_i, B_i} \|T\|^{c_j}_{A_j, B_j} \|T\|^{c_k}_{A_k, B_k}\} \tag{2}$$

where $\mathscr{P}_{(\alpha, \beta)}$ is the set of all triples (i, j, k) such that (α, β) belongs to the triangle $\overline{P_i, P_j, P_k}$ and (c_i, c_j, c_k) are the barycentric coordinates of (α, β) with respect to the points P_i, P_j, P_k.

The resulting interpolation space by these methods is only known in a few special cases. In fact in many usual cases these interpolation spaces can only be described in terms of sums and intersections of spaces, see [12] and [13]. The following is an example where the interpolated spaces by the polygons methods can be identified.

EXAMPLE 1.1. Let $\Pi = \overline{P_1 \cdots P_N}$ be our convex polygon, $P_j = (x_j, y_j)$, and for some $0 < q_1, ..., q_N \leqslant \infty$ consider the following N-tuple of scalar sequence spaces over $\mathbb{Z}$

$$(\ell_{q_1}(2^{-x_1 m - y_1 n}), \ell_{q_2}(2^{-x_2 m - y_2 n}), ..., \ell_{q_N}(2^{-x_N m - y_N n})).$$

Then for $(\alpha, \beta) \in \operatorname{Int} \Pi$ and $0 < q \leqslant \infty$ we have

$$(\ell_{q_j}(2^{-x_j m - y_j n}))_{(\alpha, \beta), q; J} = (\ell_{q_j}(2^{-x_j m - y_j n}))_{(\alpha, \beta), q; K} = \ell_q(2^{-\alpha m - \beta n})$$

with equivalence of norms.

The description of the interpolated space when we work with arbitrary weighted Lebesgue spaces is not of this simple type, see [12], [13], and [9].

Next we give a reiteration theorem relating methods associated to different polygons, see also [13]. Given a quasi-Banach N-tuple, a convex polygon Π and a point $Q=(\alpha, \beta)\in \Pi$ we say that an intermediate space X, with respect to the N-tuple $\bar{A}$, is of class $\mathscr{C}_J(Q; \bar{A})$ if $\forall x\in \Delta(\bar{A})$, $\|a\|_X \leqslant Ct^{-\alpha}s^{-\beta}J(t, s; a; \bar{A})$. The space X is of class $\mathscr{C}_K(Q; \bar{A})$ if $\forall x\in X$, $K(t, s; a; \bar{A}) \leqslant Ct^{\alpha}s^{\beta}\|a\|_X$ (where the constant C is independent of a). X is of class $\mathscr{C}(Q; \bar{A})$ iff it is of both classes, $\mathscr{C}_K(Q; \bar{A})$ and $\mathscr{C}_J(Q; \bar{A})$. For example for every interior point Q of Π, the spaces $\bar{A}_{Q,q;J}$ and $\bar{A}_{Q,q;K}$ are of class $\mathscr{C}(Q; \bar{A})$; for each vertex P_i, A_i is of class $\mathscr{C}(P_i; \bar{A})$; for every boundary point $Q\in \overline{P_i, P_{i+1}}$, $Q=(1-\theta)P_i+\theta P_{i+1}$, $(0<\theta<1)$, the space $(A_i, A_{i+1})_{\theta,q}$ is of class $\mathscr{C}(Q; \bar{A})$.

Proposition 1.2. *Let $\Pi=\overline{P_1, ..., P_N}$ be a convex polygon and $\bar{A}=\{A_1, ..., A_N\}$ a quasi-Banach N-tuple. Let $Q_1, ..., Q_M$ be M points of Π defining a convex polygon, Π' (with non empty interior), inside Π, say $Q_j=(\alpha_j, \beta_j)$. Consider M quasi-Banach spaces, $X_1, ..., X_M$, such that $X_j\in\mathscr{C}(Q_j, \bar{A})$. Let (α, β) be an interior point of the polygon $\Pi'=\overline{Q_1, ..., Q_M}$ and $0<q\leqslant\infty$, then*

$$\bar{A}^{\Pi}_{(\alpha,\beta),q;J} \hookrightarrow (X_1, ..., X_M)^{\Pi'}_{(\alpha,\beta),q;J} \hookrightarrow (X_1, ..., X_M)^{\Pi'}_{(\alpha,\beta),q;K} \hookrightarrow \bar{A}^{\Pi}_{(\alpha,\beta),q;K}.$$

Proof. We give the proof for the sake of completeness. We prove the first inclusion. $X_j\in\mathscr{C}_J(Q_j, \bar{A})$ for $1\leqslant j\leqslant M$. This implies that for all $a\in\Delta(\bar{A})$ and all $m, n\in\mathbb{Z}$

$$2^{\alpha_j m+\beta_j n}\|a\|_{X_j}\leqslant CJ(2^m, 2^n, a; \bar{A}).$$

Since $\{(\alpha_j, \beta_j)\}_{j=1}^M$ are precisely the vertices of the polygon Π' we can write

$$J'(2^m, 2^n, a; \bar{X})\leqslant CJ(2^m, 2^n, a; \bar{A}), \qquad \forall a\in\Delta(\bar{A})$$

where $J'(t, s, \cdot; \bar{X})$ stands for the J-functional associated to Π'. This gives us the continuous inclusion

$$\bar{A}^{\Pi}_{(\alpha,\beta),q;J}\hookrightarrow \bar{X}^{\Pi'}_{(\alpha,\beta),q;J}. \tag{3}$$

We turn now to the third inclusion. Let $a\in X_j$. Since $X_j\in\mathscr{C}_K(Q_j; \bar{A})$ we have the inequality

$$\sup_{m,n\in\mathbb{Z}} 2^{-\alpha_j m-\beta_j n}K(2^m, 2^n, a; \bar{A})\leqslant C\|a\|_{X_j}.$$

In particular, we have that for all $m, n \in \mathbb{Z}$, $1 \leqslant j \leqslant M$ and $a \in X_j$

$$K(2^m, 2^n, a; \bar{A}) \leqslant C 2^{\alpha_j m + \beta_j n} \|a\|_{X_j}.$$

Let $b = b_1 + \cdots + b_M \in \Sigma(\bar{X})$, then

$$K(2^m, 2^n, b; \bar{A}) \leqslant C \sum_{j=1}^{M} K(2^m, 2^n, b_j; \bar{A}) \leqslant C \sum_{j=1}^{M} 2^{\alpha_j m + \beta_j n} \|b_j\|_{X_j}.$$

Taking infimum over all possible representations of b we obtain

$$K(2^m, 2^n, b; \bar{A}) \leqslant CK'(2^m, 2^n, b; \bar{X}), \qquad \forall b \in \Sigma(\bar{X})$$

where $K'(t, s, \cdot; \bar{X})$ stands for the K-functional associated to Π'. This shows the inclusion

$$\bar{X}^{\Pi'}_{(\alpha, \beta), q; K} \hookrightarrow \bar{A}^{\Pi}_{(\alpha, \beta), q; K}. \tag{4}$$

Now inclusions (3), (4) and the inclusion $\bar{X}^{\Pi'}_{(\alpha, \beta), q; J} \hookrightarrow \bar{X}^{\Pi'}_{(\alpha, \beta), q; K}$ give the theorem. ∎

The following theorem is a straightforward application of the preceding result.

THEOREM 1.3. *Under the same hypothesis as in Proposition* 1.2 *it yields that if the N-tuple $\bar{A}$ satisfies the J-K equivalence property we have*

$$\bar{A}^{\Pi}_{(\alpha, \beta), q} = \bar{X}^{\Pi'}_{(\alpha, \beta), q}$$

Recently I. U. Asekritova and N. Ya. Krugljak have proved that (the multidimensional) Sparr J- and K-methods have the equivalence property when they deal with function lattices, see [1].

The K and J methods associated to polygons can be described by maximal and minimal interpolation functors in the sense of Aronszajn–Gagliardo, see [2]. When we restrict ourselves to Banach N-tuples this was done by Cobos and Peetre in [7]. However, if we deal with quasi-Banach tuples the maximal description of the K-method does not work since it relies on the Hahn-Banach Theorem. Recently Bergh and Cobos gave a maximal description for the classical real method for couples in the category of quasi-Banach spaces, see [4]. Following them we outline this result in the case of the polygons K-method.

Let T be a mapping from a quasi-Banach space A into the scalar sequence space $\mathbb{R}^I_+$ where I stand for some set of indexes. We say T is quasi-sublinear (QSL) with constant $C \geqslant 1$ if

(i) $T(a+b) \leqslant C(T(a)+T(b))$, $a, b \in A$

(ii) $T(\lambda a) = \lambda T(a)$, $\lambda \in \mathbb{R}_+$, $a \in A$.

(the inequalities must be understood coordinatewise).

Let $\bar{A}$ be a quasi-Banach N-tuple with quasi-norm constant C ($C = \max\{C_i\}$ where C_i stands for the quasi-norm constant of the space A_i). Let $\Pi = \overline{P_1, ..., P_N}$ be our convex polygon with vertices $P_j = (x_j, y_j)$. Consider the N-tuple

$$\bar{\ell}_\infty = \{\ell_\infty(2^{x_1 m + y_1 n}), \ell_\infty(2^{x_2 m + y_2 n}), ..., \ell_\infty(2^{x_N m + y_N n})\}.$$

Given $C \geqslant 1$ we define $\mathcal{Q}_C(\bar{A}, \bar{\ell}_\infty)$ as the collection of all those quasi-sublinear operators $T: \Sigma(\bar{A}) \to \mathbb{R}_+^{\mathbb{Z}}$ whose restrictions to A_j, $1 \leqslant j \leqslant N$ define quasi-sublinear operators $T: A_j \to \ell_\infty(2^{mx_j + ny_j})$ with quasi-sublinear constant C. We put

$$\|T\|_{\bar{A}, \bar{\ell}_\infty} = \max\{\|T\|_{A_i, \ell_\infty(2^{mx_i + ny_i})}; i: 1, ..., N\}.$$

Given $(\alpha, \beta) \in \operatorname{Int} \Pi$ and $0 < q \leqslant \infty$ we define the space

$$\tilde{H}_C[\bar{\ell}_\infty; \ell_q(2^{-\alpha m - \beta n})](\bar{A})$$

as the collection of all those $a \in \Sigma(\bar{A})$ such that $T(a) \in \ell_q(2^{-\alpha m - \beta n})$ for any $T \in \mathcal{Q}_C(\bar{A}, \bar{\ell}_\infty)$. We also define the quasi-norm

$$\|a\|_{\tilde{H}_C} = \sup\{\|Ta\|_{\ell_q(2^{-\alpha m - \beta n})}; T \in \mathcal{Q}_C(\bar{A}, \bar{\ell}_\infty), \|T\|_{\bar{A}, \bar{\ell}_\infty} \leqslant 1\}.$$

The description of the K-method as a maximal method is given by the next result.

Proposition 1.4. *Let $\Pi =$ be a convex polygon and let $\bar{A}$ be a quasi-Banach N-tuple with quasi-norm constant C. For every $(\alpha, \beta) \in \operatorname{Int} \Pi$ and $0 < q \leqslant \infty$ we have that*

$$\tilde{H}_C[\bar{\ell}_\infty; \ell_q(2^{-\alpha m - \beta n})](\bar{A}) = \bar{A}^{\Pi}_{(\alpha, \beta), q; K}$$

with equivalence of quasi-norms.

The proof is similar to that in [4].

Recall that if $(A, \|\cdot\|_A)$ is a quasi-Banach space with constant C, then $\|\cdot\|_A$ is equivalent to a p-norm with p verifying that $(2C)^p = 2$. For this reason we can consider our quasi-Banach N-tuple as a p-Banach N-tuple.

The J-method associated to polygons can be described as a minimal interpolation method in the category of quasi-Banach spaces. In the case of the classical real method for couples this was done by Mastylo, see [17] and [11]. We adapt Mastylo's description to the N-tuples case.

Let $\bar{X}$ be a quasi-Banach N-tuple such that the dual of quasi-Banach space $\sum(\bar{X})$ is separating (total in the terminology of [17]) and let X be a non-trivial quasi-normed space such that $X \hookrightarrow \sum(\bar{X})$. Fix some $0 < p \leqslant 1$. Then for any quasi-Banach tuple $\bar{A}$ define the p-orbit space $\mathscr{G}_p(\bar{X}; X)(\bar{A})$ as the set of all elements $a \in \sum(\bar{A})$ for which there exists a representation of the form

$$a = \sum_{n=1}^{\infty} T_n x_n$$

with convergence in $\Sigma(\bar{A})$, where $T_n \in \mathscr{L}(\bar{X}, \bar{A})$, $x_n \in X$ and

$$\left(\sum_1^{\infty} (\|T_n\|_{\bar{X}, \bar{A}} \|x_n\|_X)^p\right)^{1/p} < \infty.$$

This space is endowed with the norm

$$\|a\|_{\mathscr{G}_p}(\bar{X}; X)(\bar{A}) = \inf\left\{\left(\sum_1^{\infty} (\|T_n\|_{\bar{X}, \bar{A}} \|x_n\|_X)^p\right)^{1/p}\right\}$$

where the infimum extends over all representations of a as above. Clearly $\|\cdot\|_{\mathscr{G}_p(\bar{X}; X)(\bar{A})}$ is a p-norm on the space $\mathscr{G}_p(\bar{X}; X)(\bar{A})$ and $\mathscr{G}_p(\bar{X}; X)(\cdot)$ is an interpolation functor on the class of all p-Banach tuples, see [17].

Associated to the polygon $\Pi = \overline{P_1, ..., P_N}$ consider the following tuple of sequences spaces

$$\bar{\ell}_p = \{\ell_p(2^{-x_1 m - y_1 n}), \ell_p(2^{-x_2 m - y_2 n}), ..., \ell_p(2^{-x_N m - y_N n})\}. \tag{5}$$

With the help of this tuple we characterize the J method associated to Π as a minimal interpolation method.

Proposition 1.5. *Let $\bar{A}$ be a p-Banach space N-tuple, let $(\alpha, \beta) \in \operatorname{Int} \Pi$ and $0 < q \leqslant \infty$. Put $r = \min\{p, q\}$, then*

$$\bar{A}_{(\alpha, \beta), q; J} = \mathscr{G}_r[\bar{\ell}_p; \ell_q(2^{-\alpha m - \beta n})](\bar{A})$$

with equivalence of norms.

The proof follows the lines of that in [17] for the real method for couples.

2. WOLFF THEOREM

In order to establish Wolff Theorem we need some preliminaries. First of all we extend estimates (1) and (2) for norm of the interpolated operator to the case of quasi-sublinear operator between K-spaces.

PROPOSITION 2.1. *Let $\bar{X}$ be a quasi-Banach N-tuple and $\bar{Y}$ be a quasi-Banach lattice N-tuple. Then every bounded QSL operator $T: \bar{X} \to \bar{Y}$ is also a bounded operator $T: \bar{X}_{Q,p;K} \to \bar{Y}_{Q,p;K}$. Moreover, if we put $M_i = \|T\|_{X_i, Y_i}$ for $1 \leqslant i \leqslant N$ then we have the inequality*

$$\|T: \bar{X}_{Q,p;K} \to \bar{Y}_{Q,p;K}\| \leqslant C^{N-1} \max_{(i,j,k) \in \mathscr{P}_Q} \{M_i^{c_i} M_j^{c_j} M_k^{c_k}\} \tag{6}$$

where $\mathscr{P}_Q$ is the set of all triples (i, j, k) such that Q belongs to the triangle $\overline{P_i, P_j, P_k}$ and (c_i, c_j, c_k) are the barycentric coordinates of Q with respect to the points P_i, P_j, P_k and C is the quasi-sublinearity constant of T.

Proof. It suffices to show that for every $t, s > 0$, $\lambda, \mu > 0$ and $x \in \Sigma(\bar{X})$ we have that

$$K(t, s, Ta; \bar{Y}) \leqslant C^{N-1} \max_{1 \leqslant j \leqslant N} \{\lambda^{x_j} \mu^{y_j} M_j\} \; K\left(\frac{t}{\lambda}, \frac{s}{\mu}, a; \bar{X}\right).$$

If $x = \sum_{i=1}^N x_i$ with $x_i \in X_i$ we have that $T(x) \leqslant C^{N-1} \sum_{i=1}^N T(x_i)$ which implies that for $1 \leqslant i \leqslant N$ there exists $y_i \in Y_i$, verifying that $0 \leqslant y_i \leqslant T(x_i)$ with $T(x) = C^{N-1} \sum_{i=1}^N y_i$. Now

$$\begin{aligned}
K(t, s, T(x); \bar{Y}) &\leqslant C^{N-1} \sum_{i=1}^N \lambda^{x_i} \mu^{y_i} \left(\frac{t}{\lambda}\right)^{x_i} \left(\frac{s}{\mu}\right)^{y_i} \|y_i\|_{Y_i} \\
&\leqslant C^{N-1} \sum_{i=1}^N \lambda^{x_i} \mu^{y_i} \left(\frac{t}{\lambda}\right)^{x_i} \left(\frac{s}{\mu}\right)^{y_i} \|T\|_{X_i, Y_i} \|x_i\|_{X_i} \\
&\leqslant C^{N-1} \max_{1 \leqslant i \leqslant N} \{\lambda^{x_i} \mu^{y_i} M_i\} \sum_{i=1}^N \left(\frac{t}{\lambda}\right)^{x_i} \left(\frac{s}{\mu}\right)^{y_i} \|x_i\|_{X_i}.
\end{aligned}$$

This gives the desired inequality. Now follow the proof given by Cobos, Schonbek and one of the present authors in [9] to conclude the result. ∎

We introduce the notion of F-interpolation space.

DEFINITION 2.2. Let $\bar{A} = \{A_1, ..., A_N\}$ and $\bar{B} = \{B_1, ..., B_N\}$ be two interpolation N-tuples and let $\bar{X} = \{X_1, ..., X_M\}$, $\bar{Y} = \{Y_1, ..., Y_M\}$ be two M-tuples of intermediate spaces with respect to $\bar{A}$ and $\bar{B}$ respectively. For every $k = 1, ..., M$, let $\varnothing \neq I_k \subseteq \{1, ..., N\}$ and $J_k \subseteq \{1, ..., M\}$. Assume that for $k=1, ..., M$, the spaces X_k and Y_k are interpolation spaces with respect to the tuples $((A_i)_{i \in I_k}, (X_j)_{j \in J_k})$ and $((B_i)_{i \in I_k}, (Y_j)_{j \in J_k})$.

We say that X_k and Y_k are F_k relative interpolation spaces if for every operator $T: \bar{A} \to \bar{B}$ which acts also $\bar{X} \to \bar{Y}$ we have

$$\|T: X_k \to Y_k\| \leqslant F_k(\max_{i \in I_k} \{\|T\|_{A_i, B_i}\}, \max_{j \in J_k} \{\|T\|_{X_j, Y_j}\}) \tag{7}$$

where $F_k: \mathbb{R}_+^2 \to \mathbb{R}_+$ is an homogeneous non-decreasing positive real valued function of two variables verifying that $\lim_{u \to 0} F_k(u, 1) = 0$.

Now we set conditions to state Wolff Theorem. Let $\Pi = \overline{P_1, ..., P_N}$ be a convex polygon and let $\bar{A} = \{A_1, ..., A_N\}$ a quasi-Banach space N-tuple. Let $\{X_1, ..., X_M\}$ be M intermediate spaces with respect to the N-tuple $\bar{A}$ and $Q_1, ..., Q_M$ be M points in Int Π. We can imagine each space X_k as sitting on the point Q_k. For each $1 \leqslant k \leqslant M$ consider a convex polygon Π_k, whose vertices are in the union $\{P_1, ..., P_N\} \cup \{Q_1, ..., Q_M\}$, verifying that $Q_k \in \text{Int}\, \Pi_k$. Each polygon Π_k defines two sets of indices, $I_k \subset \{1, ..., N\}$ and $J_k \subset \{1, ..., M\}$. An index $j \in I_k$ if P_j is a vertex of Π_k. Similarly, $j \in J_k$ if Q_j is a vertex of Π_k. We also assume that each polygon Π_k has at least one vertex in the set $\{P_1, ..., P_N\}$, i.e., $I_k \neq \emptyset$.

Proposition 2.3. *Let $\bar{A} = \{A_1, ..., A_N\}$ and $\bar{B} = \{B_1, ..., B_N\}$ be two quasi-Banach N-tuples and let $\bar{X} = \{X_1, ..., X_M\}$, $\bar{Y} = \{Y_1, ..., Y_M\}$ as in Definition 2.2. Assume that for some $0 < q_j \leqslant \infty$, $1 \leqslant j \leqslant M$*

$$X_k \hookrightarrow (\bar{A}_{I_k}, \bar{X}_{J_k})^{\Pi_k}_{Q_k, q_k} \qquad \text{and} \qquad (\bar{B}_{I_k}, \bar{Y}_{J_k})^{\Pi_k}_{Q_k, q_k} \hookrightarrow Y_k$$

(both spaces are obtained by the same K or J-method). Then X_k and Y_k are F_k relative interpolation spaces with respect to $\bar{A}$ and $\bar{B}$. Further, even if we consider QSL operators $T: \bar{A} \to \bar{B}$, where $\bar{B}$ is a quasi-Banach lattice N-tuple, then the above inequality (7) for the norm of the interpolated operator by K method holds.

Proof. We restrict first to linear operators. Put $\Pi_k = \overline{S_1, ..., S_{M_k}}$ the explicit vertices of Π_k, the involved tuples being

$$(A_{I_k}, X_{J_k}) = \{Z_1, ..., Z_{M_k}\}$$

$$(B_{I_k}, Y_{J_k}) = \{U_1, ..., U_{M_k}\}$$

Then if $T: \Sigma(\bar{A}) \to \Delta(\bar{B})$ we have the following estimate for the norm of the interpolated operator,

$$\|T: X_k \to Y_k\| \leqslant \max\{\|T\|^{c_r}_{Z_r, U_r} \|T\|^{c_s}_{Z_s, U_s} \|T\|^{c_t}_{Z_t, U_t}\} \tag{8}$$

where the maximum extends over all triples (r, s, t) such that Q_k lays in the triangle $\overline{S_r, S_s, S_t}$ and (c_r, c_s, c_t) are the barycentric coordinates of Q_k with respect to the vertices of the triangle (note that at least two of these coordinates are nonzero).

If $\{Q_1, ..., Q_M\}$ are the vertices of a convex polygon, each triple $(S_r, S_s, S_t) = \mathcal{T}$ contains at least one vertex P_i of Π. Let $\alpha(k, \mathcal{T})$ be the sum of the barycentric coordinates of Q_k relative to those vertices of $\mathcal{T}$ which are vertices of Π, $\alpha(k, \mathcal{T})$ is positive, and hence $\alpha(k) = \min_{\mathcal{T}} \{\alpha(k, \mathcal{T})\}$ is positive. So the function

$$F_k(u, v) = u^{\alpha(k)} v^{1-\alpha(k)}$$

makes X_k and $Y_k F_k$ relative interpolation spaces.

In case $\{Q_1, ..., Q_M\}$ are not the vertices of a convex polygon we have to use a different argument. Choose a triangle $\mathcal{T}_1 = \overline{S_{1,1}, S_{1,2}, S_{1,3}}$ (or $\mathcal{T}_1 = \overline{S_{1,1}, S_{1,2}}$ if Q_k lays on a side of the triangle) containing Q_k and realizing the maximum of inequality (8). If some vertex of $\mathcal{T}_1$ is a vertex of Π we repeat the above argument, if not, all the points $S_{1,r}$, $r = 1, 2, (3)$, are of the form $Q_{1,r} \in \{Q_1, ..., Q_M\}$. We repeat the procedure for these points. Choose triangles $\mathcal{T}_{1,1}, \mathcal{T}_{1,2}, (\mathcal{T}_{1,3})$ such that $Q_{1,r} \in \mathcal{T}_{1,r}$ for $r = 1, 2, (3)$ and realizing the maximum of inequality (8) i.e.,

$$\begin{aligned} \|T\|_{X_{1,r}, Y_{1,r}} &\leqslant \max\{ \|T\|^{c_u}_{Z_u, U_u} \|T\|^{c_v}_{Z_v, U_v} \|T\|^{c_w}_{Z_w, U_w} \} \\ &= \|T\|^{c_1(\mathcal{T}_{1,r})}_{X_{1,r,1}, Y_{1,r,1}} \|T\|^{c_2(\mathcal{T}_{1,r})}_{X_{1,r,2}, Y_{1,r,2}} \|T\|^{c_3(\mathcal{T}_{1,r})}_{X_{1,r,3}, Y_{1,r,3}} \end{aligned}$$

(here the maximum extends over all triples (u, v, w) such that the point $Q_{1,r}$ lays in the triangle $\overline{S_u, S_v, S_w}$). If the vertices $S_{1,r,j}$, for $r = 1, 2, (3)$ and $j = 1, 2, (3)$, all belong to $\{Q_1, ..., Q_M\}$ we repeat the process and define a new generation of triangles $\mathcal{T}_{1,r,j}$ with $r = 1, 2, (3)$ and $j = 1, 2, (3)$. This process ends at the mth generation if one of the triangles of this generation has one of the P_i's as a vertex (with nonzero corresponding barycentric coordinate). Thus we define a martingale $(S_{i_1, ..., i_m})$ in the plane with $m \geqslant 1$ and $i_j = 1, 2, (3)$. For each index, $(i_1, ..., i_m)$ we have a system of positive coefficients $c_{i_1, ..., i_m} > 0$ verifying that

$$\sum_{j=1,2,(3)} c_{i_1, ..., i_{m-1}, j} = 1 \qquad \text{and}$$

$$\sum_{j=1,2,(3)} c_{i_1, ..., i_{m-1}, j} S_{i_1, ..., i_{m-1}, j} = S_{i_1, ..., i_{m-1}}.$$

Since the set of the Q_j's and the P_i's is finite,

$$0 < \varepsilon = \inf\{ \|S' - S''\|; S' \neq S'' \in \{Q_j\text{'s}\} \cup \{P_i\text{'s}\}\}$$

and so

$$\inf_{m;\, i_1, \ldots, i_m} \{ \| S_{i_1, \ldots, i_{m-1}} - S_{i_1, \ldots, i_m} \| \} \geqslant \varepsilon.$$

As a consequence the maximal number, $\hat{m}$, of generations of the martingale $(S_{i_1, \ldots, i_m})$ is bounded by $1/\varepsilon^2 (\operatorname{diam} \Pi)^2$. To see this, fix P a vertex of Π. We have

$$\begin{aligned}
&\sum_{j=1,2,(3)} c_{i_1, \ldots, i_{m-1}, j} \| P - S_{i_1, \ldots, i_{m-1}, j} \|^2 \\
&\quad = \| P - S_{i_1, \ldots, i_{m-1}} \|^2 \\
&\qquad + \sum_{j=1,2,(3)} c_{i_1, \ldots, i_{m-1}, j} \| S_{i_1, \ldots, i_{m-1}, j} \| S_{i_1, \ldots, i_{m-1}} - S_{i_1, \ldots, i_m} \|^2 \\
&\quad \geqslant \| P - S_{i_1, \ldots, i_{m-1}} \|^2 + \varepsilon^2.
\end{aligned}$$

Hence

$$\sup_{i_1, \ldots, i_m} \| P - S_{i_1, \ldots, i_m} \|^2 \geqslant \sup_{i_1, \ldots, i_{m-1}} \| P - S_{i_1, \ldots, i_{m-1}} \|^2 + \varepsilon^2 \geqslant \cdots \geqslant m\varepsilon^2.$$

Since P is a vertex of Π and $S_{i_1, \ldots, i_m} \in \{ Q_j\text{'s} \} \cup \{ P_i\text{'s} \}$ we have the inequality

$$(\operatorname{diam} \Pi)^2 \geqslant m\varepsilon^2.$$

So the process is finite and the last generation of the martingale $(S_{i_1, \ldots, i_{\hat{m}}})$ contains at least one vertex P_i of Π. For $m < \hat{m}$ we have that for some j $S_{i_1, \ldots, i_m} = Q_j$. For simplicity put $M_j = \| T \colon X_j \to Y_j \|$. Using inequality (9) recursively we have for $m < \hat{m}$

$$\begin{aligned}
M_k &\leqslant M_1^{c_1} M_2^{c_2} M_3^{c_3} \\
&\leqslant (M_{1,1}^{c_{1,1}} M_{1,2}^{c_{1,2}} M_{1,3}^{c_{1,3}})^{c_1} (M_{2,1}^{c_{2,1}} M_{2,2}^{c_{2,2}} M_{2,3}^{c_{2,3}})^{c_2} (M_{3,1}^{c_{3,1}} M_{3,2}^{c_{3,2}} M_{3,3}^{c_{3,3}})^{c_3} \\
&\leqslant \cdots \leqslant \prod_{i_1, \ldots, i_m} M_{i_1, \ldots, i_m}^{\gamma_{i_1, \ldots, i_m}}
\end{aligned}$$

where $\gamma_{i_1, \ldots, i_m} = c_{i_1} c_{i_1, i_2} \cdots c_{i_1, \ldots, i_m} > 0$ and $\sum_{1 \leqslant m < \hat{m};\, i_1, \ldots, i_m} \gamma_{i_1, \ldots, i_m} = 1$. Assume that $P_i = S_{i_1, \ldots, i_{\hat{m}-1}, 1}$, then

$$M_{i_1, \ldots, i_{\hat{m}-1}} \leqslant \| T \colon A_i \to B_i \|^{c_{i_1, \ldots, i_{\hat{m}}}} \max\{ \| T \|_{\bar{A}, \bar{B}}, \| T \|_{\bar{X}, \bar{Y}} \}^{1 - c_{i_1, \ldots, i_{\hat{m}}}}.$$

Hence

$$
\begin{aligned}
M_k &\leqslant [\,\|T\colon A_i \to B_i\|^{c_{i_1, ..., i_{\hat{m}}}} \\
&\qquad \times \max\{\|T\|_{\bar{A}, \bar{B}}, \|T\|_{\bar{X}, \bar{Y}}\}^{1-c_{i_1, ..., i_{\hat{m}}}}]^{\gamma_{i_1, ..., i_{\hat{m}-1}}} \|T\|_{\bar{X}, \bar{Y}}^{1-\gamma_{i_1, ..., i_{\hat{m}-1}}} \\
&= \|T\colon A_i \to B_i\|^{\gamma_{i_1, ..., i_{\hat{m}-1}, 1}} \\
&\qquad \times \max\{\|T\|_{\bar{A}, \bar{B}}, \|T\|_{\bar{X}, \bar{Y}}\}^{(1-c_{i_1, ..., i_{\hat{m}-1}, 1})\, \gamma_{i_1, ..., i_{\hat{m}-1}}} \|T\|_{\bar{X}, \bar{Y}}^{1-\gamma_{i_1, ..., i_{\hat{m}-1}}} \\
&\leqslant \|T\colon A_i \to B_i\|^{\gamma_{i_1, ..., i_{\hat{m}-1}, 1}} \max\{\|T\|_{\bar{A}, \bar{B}}, \|T\|_{\bar{X}, \bar{Y}}\}^{1-\gamma_{i_1, ..., i_{\hat{m}-1}, 1}}.
\end{aligned}
$$

Let δ be the least nonzero barycentric coordinate of all points Q_j's with respect to a triangle $\mathscr{T}$ with vertices in $\{Q_k\text{'s}\} \cup \{P_i\text{'s}\}$ ($\delta > 0$ since it is the minimum of finite set of positive numbers). Then, $\gamma_{i_1, ..., i_{\hat{m}}} \geqslant \delta^{\hat{m}} \geqslant \delta^{(\operatorname{diam} \Pi/\varepsilon)} = \delta_0 > 0$. Hence

$$M_k \leqslant \|T\|_{\bar{A}, \bar{B}}^{\delta_0} \max\{\|T\|_{\bar{A}, \bar{B}}, \|T\|_{\bar{X}, \bar{Y}}\}^{1-\delta_0}$$

and the functions

$$F_k(u, v) = u^{\delta_0} \max\{u, v\}^{1-\delta_0}$$

makes all the spaces X_k's and Y_k's F_k interpolation spaces.

The proof for the K-method when we deal with QSL operators acting from a quasi-Banach N-tuple into a quasi-Banach lattice N-tuple is completely similar. ▮

The following lemma can be found in [6], Lemma 1.3.

Lemma 2.4. *Assume that X_k and Y_k are F_k relative interpolation spaces for $k = 1, ..., M$. Then every operator $T\colon \Sigma(\bar{A}) \to \Delta(\bar{B})$ acts from $\Sigma(X) \to \Delta(Y)$ and*

$$\max_{1 \leqslant k \leqslant M} \{\|T\colon X_k \to Y_k\|\} \leqslant C\, \|T\|_{\bar{A}, \bar{B}}.$$

Theorem 2.5. *Assume that for $1 \leqslant j \leqslant M$ and $1 \leqslant q_1, q_2, ..., q_M \leqslant \infty$*

$$X_j = (\bar{A}_{I_j}, \bar{X}_{J_j})^{\Pi_j}_{Q_j, q_j; J} \qquad or \qquad X_j = (\bar{A}_{I_j}, \bar{X}_{J_j})^{\Pi_j}_{Q_j, q_j; K}.$$

Then if $\bar{A}$ satisfies the J-K methods equivalence property we have for $1 \leqslant j \leqslant M$

$$X_j = \bar{A}^{\Pi}_{Q_j, q_j}.$$

Proof. Let $0<p\leqslant 1$ such that all the involved spaces are p-Banach spaces and $p<q_1, ..., q_M$. Using the description of the J-method as a minimal interpolation functor we have the following inclusions

$$\mathcal{G}_p[\bar{\ell}_p(k), \ell_{q_k}(2^{-\langle (m,n), Q_k\rangle})](\bar{A}_{I_k}, \bar{X}_{J_k}) = (\bar{A}_{I_k}, \bar{X}_{J_k})^{\Pi_k}_{Q_k, q_k; J} \hookrightarrow X_k$$

where the tuples $\bar{\ell}_p(k)$ are the corresponding sequence spaces associated to the polygons Π_k as in equality (5). On the other hand

$$\begin{aligned}\mathcal{G}_p[\bar{\ell}_p(k), \ell_{q_k}(2^{-\langle (m,n), Q_k\rangle})](\bar{\ell}_p(k)) &= (\bar{\ell}_p(k))^{\Pi_k}_{Q_k, q_k; J}\\ &= \ell_{q_k}(2^{-\langle (m,n), Q_k\rangle})\end{aligned}$$

(by Example 1.1)

By Proposition 2.3 $\ell_{q_k}(2^{-\langle (m,n), Q_k\rangle})$ and X_k are F_k relative interpolation spaces, relatively to $\bar{\ell}_p$ and $\bar{A}$. Then apply Lemma 2.4 and follow [6] to obtain the inclusion

$$\mathcal{G}_p[\bar{\ell}_p, \ell_{q_k}(2^{-\langle (m,n), Q_k\rangle})](\bar{A}) \hookrightarrow X_k.$$

Now choose $C\geqslant 1$ such that all the spaces involved are quasi-Banach spaces with constant C. From the hypothesis we have the inclusions

$$X_k \hookrightarrow \tilde{H}_C[\bar{\ell}_\infty(k); \ell_{q_k}(2^{-\langle Q_k, (m,n)\rangle})](\bar{A}_{I_k}, \bar{X}_{J_k})$$

for $1\leqslant k\leqslant M$. Taking into account that $\bar{\ell}_\infty$ is a Banach lattice N-tuple and that the spaces X_k and $\ell_{q_k}(2^{-\langle (m,n), Q_k\rangle})$ are F_k interpolation spaces, even if we deal with QSL operators $T\colon \bar{A}\to\bar{\ell}_\infty$, we can apply Lemma 2.4 and proceed as in [6] to show

$$X_k \hookrightarrow \tilde{H}_C[\bar{\ell}_\infty; \ell_{q_k}(2^{-\langle Q_k, (m,n)\rangle})](\bar{A}).$$

Now use these inclusions and the equalities of the hypotheses

$$X_j = (\bar{A}_{I_j}, \bar{X}_{J_j})^{\Pi_j}_{Q_j, q_j; J} \qquad \text{or} \qquad X_j = (\bar{A}_{I_j}, \bar{X}_{J_j})^{\Pi_j}_{Q_j, q_j; K}$$

together with reiteration theorem to conclude the proof. ∎

3. INTERPOLATION SCALES FOR THE POLYGONS METHOD

We adapt the definition of interpolation scales given by Cobos and Peetre in [11] to the polygons methods.

DEFINITION 3.1. Let Ω be an open subset of $\mathbb{R}^2$. Let $\varphi: \Omega \to \mathbb{R}$ be an $(0, \infty]$ valued function on Ω such that $1/\varphi$ is affine. In particular $1/\varphi = (1-\theta)/\varphi(x_1) + \theta/\varphi(x_2)$ whenever $x, x_1, x_2 \in \Omega, 0 \leqslant \theta \leqslant 1$ and $x = (1-\theta) x_1 + \theta x_2$.

Let $\mathscr{U}$ be a topological linear Hausdorff space. By an interpolation scale over Ω, contained in $\mathscr{U}$, we mean a family of quasi-Banach spaces $\{A_x\}_{x \in \Omega}$, all continuously embedded in $\mathscr{U}$, which is closed for interpolation, i.e.,

$$A_x = (A_{x_1}, A_{x_2}, ..., A_{x_N})_{x, \varphi(x)}$$

whenever $x, x_1, x_2, ..., x_N \in \Omega$, $\{x_1, x_2, ..., x_N\}$ form a convex polygon and $x \in \operatorname{Int} \operatorname{conv}\{x_1, ..., x_N\}$.

EXAMPLE 3.2. Let $\Pi = \overline{P_1, ..., P_N}$ be a convex polygon in $\mathbb{R}^2$ and let $\bar{A} = \{A_1, ..., A_N\}$ be a quasi-Banach N-tuple satisfying the J-K equivalence property. If $1/\varphi: \operatorname{Int} \Pi \to [0, \infty)$ is an affine mapping, then

$$\{\bar{A}_{(\alpha, \beta), \varphi(\alpha, \beta)}\}_{(\alpha, \beta) \in \operatorname{Int} \Pi}$$

is an interpolation scale.

Proof. Let $x, x_1, ..., x_M \in \operatorname{Int} \Pi$, $x_1, ..., x_M$ forming a convex polygon and $x \in \operatorname{Int} \operatorname{conv}\{x_1, ..., x_M\}$. Then the reiteration result, Theorem 1.3 shows that

$$(A_{x_1}, ..., A_{x_M})_{x, \varphi(x)} = \bar{A}_{x, \varphi(x)} = A_x. \quad \blacksquare$$

Subsequently we shall deal with interpolation scales associated to polygons. J- and K-methods agree as in the example. We consider the following question asked in [6]. Can overlapping interpolation scales be pasted together?

Let $\{\bar{A}_{(\alpha, \beta), \varphi(\alpha, \beta)}\}_{(\alpha, \beta) \in \operatorname{Int} \Pi_1}$, be the interpolation scale associated to the N-tuple $\bar{A}$, the polygon Π_1 and the affine function φ. Let $\{\bar{B}_{(\alpha, \beta), \phi(\alpha, \beta)}\}_{(\alpha, \beta) \in \operatorname{Int} \Pi_2}$ be the interpolation scale associated to $\bar{B}$, Π_2 and the affine function ϕ. We say that both scales overlap if:

1. Both scales are embedded in the same linear Hausdorff space.
2. $\operatorname{Int} \Pi_1 \cap \operatorname{Int} \Pi_2 \neq \varnothing$
3. $\bar{A}_{(\alpha, \beta), \varphi(\alpha, \beta)} = \bar{B}_{(\alpha, \beta), \phi(\alpha, \beta)}$, $\forall (\alpha, \beta) \in \operatorname{Int} \Pi_1 \cap \operatorname{Int} \Pi_2$
4. The spaces agree on the vertices, i.e., If $\Pi_2 = \overline{Q_1, ..., Q_M}$ and $Q_i \in \operatorname{Int} \Pi_1$, then $B_i = \bar{A}_{Q_i, \varphi(Q_i)}$. Similarly for the vertices of Π_1, if $Q_i \in \{P_1, ..., P_N\}$, say $Q_i = P_j$, then $B_i = A_j$.

Clearly, if the scale A and B overlap, the functions φ and ϕ coincide on an open set of $\mathbb{R}^2$, (see [10] and note we can assume without loss of generality that the intersections are not closed in the corresponding sums). Thus, $1/\varphi$ and $1/\phi$ agree on an open set of $\mathbb{R}^2$. Since the latter are affine functions, they agree on $\mathbb{R}^2$ and are $[0, \infty)$ valued on the convex hull of Π_1 and Π_2.

Assume that the scales A and B overlap. We will find an interpolation scale containing both scales. Let Π be the convex polygon generated by all vertices of Π_1 and Π_2,

$$\Pi = \text{conv}\{P_1, ..., P_N, Q_1, ..., Q_M\} = \overline{R_1, ..., R_t}.$$

Let us define the tuple $\bar{C} = \{C_1, ..., C_t\}$ where the C_j is the space corresponding to the vertex R_j. Assume that all the spaces of $\bar{A}$ and $\bar{B}$ are intermediate spaces with respect to $\bar{C}$ and that the tuple $\bar{C}$ verifies the *J-K* equivalence property. Consider the interpolation scale

$$C = \{\bar{C}_{(\alpha, \beta), \varphi(\alpha, \beta)}\}_{(\alpha, \beta) \in \text{Int}\, \Pi}$$

PROPOSITION 3.3. *If all vertices of Π_1 and Π_2 (all the P's and the Q's) are either on the vertices of Π or in the intersection of Π_1 and Π_2 then the interpolation scale C pastes together the scales A and B; further, C fills in the gaps between the two polygons Π_1 and Π_2.*

Proof. (a) Assume first that if Q_j is not a vertex of Π, then $Q_j \in \text{Int}\, \Pi_1$, and similarly, if some vertex of Π_1, say P_i, is not a vertex of Π, then $P_i \in \text{Int}\, \Pi_2$.

An easy application of Wolff theorem shows that all the points in the intersection (including the vertices) are generated by the *N*-tuple associated to Π, see Fig. 1.

Now let $(\alpha, \beta) \in \text{Int}\, \Pi_1$ (or $(\alpha, \beta) \in \text{Int}\, \Pi_2$). We can choose a triangle $\mathcal{T}$ with vertices in $\Pi_1 \subset \Pi$ and in the intersection $\text{Int}\, \Pi_1 \cap \text{Int}\, \Pi_2$ such that

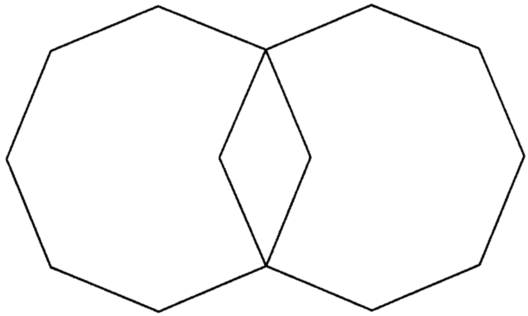

FIGURE 1

$(\alpha, \beta) \in \operatorname{Int} \mathcal{T}$. A straightforward application of the reiteration theorem shows that

$$\bar{A}_{(\alpha, \beta), \varphi(\alpha, \beta)} = \bar{\mathcal{T}}_{(\alpha, \beta), \varphi(\alpha, \beta)} = \bar{C}_{(\alpha, \beta), \varphi(\alpha, \beta)}$$

(b) *General case.* Allow some vertices of Π_1 to lie on the boundary of Π_2 and viceversa. We distinguish three possible kinds of these points, say $P_i \in \partial \Pi_2$.

1. $P_i \in (Q_j, Q_{j+1})$ and P_{i-1}, P_{i+1} are separated by the line (Q_j, Q_{j+1}).
2. $P_i \in (Q_j, Q_{j+1})$ and P_{i-1}, P_{i+1} are in the same half-plane determined by the line (Q_j, Q_{j+1}).
3. $P_i = Q_j$ for some j.

We aim to rid of these types of points and reduce the problem to *a*).

First we deal with the points of the 1st kind. Here one of the points P_{i-1}, P_{i+1} (say P_{i-1}) belongs to the same half-plane as $\operatorname{Int} \Pi_1 \cap \operatorname{Int} \Pi_2$ with respect to the line (Q_j, Q_{j+1}). Choose $P_i' \in (P_i, P_{i-1})$ close enough to P_i so that the line (P_i', P_i) does not contain any vertex of Π_2. Clearly P_i' belongs to $\operatorname{Int} \Pi_2$. The modified polygon $\Pi_1' = (P_1, ..., P_i', ..., P_N)$ still contains all the vertices of Π_2 which where in Π_1, see Fig. 2.

We eliminate in this way successively all points of first kind of Π_1, and then those of Π_2.

Then similar arguments, according to the corresponding figures, (Figs. 3 and 4), show us how to modify the polygons in order to rid of the points of the second and the third kind.

Iterating these processes for both Π_1 and Π_2 we find alterations, $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$, of the polygons Π_1 and Π_2 resp., verifying that

1. $\tilde{\Pi}_1 \subseteq \Pi_1$, $\tilde{\Pi}_2 \subseteq \Pi_2$.
2. $\operatorname{conv}(\tilde{\Pi}_1 \cup \tilde{\Pi}_2)) = \operatorname{conv}(\Pi_1 \cup \Pi_2)$.

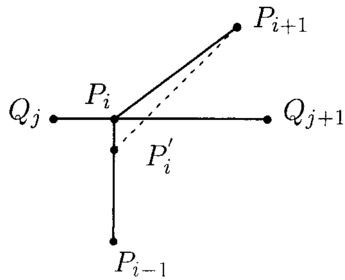

FIG. 2. Points of the 1st kind.

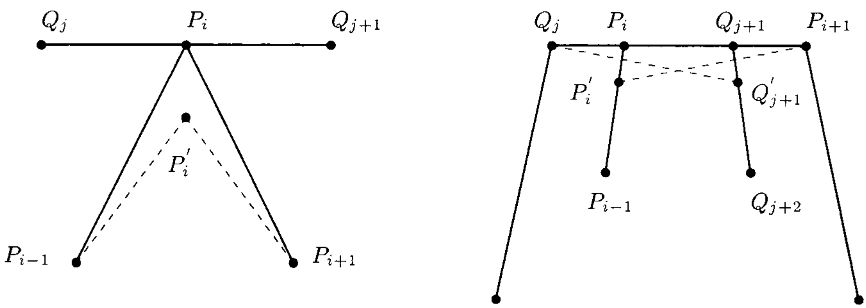

FIG. 3. Points of the 2nd kind.

3. Int $\tilde{\Pi}_1$ contains all the vertices of $\tilde{\Pi}_2$ not in Π, and Int $\tilde{\Pi}_2$ contains all the vertices of $\tilde{\Pi}_1$ not in Π.

Apply (a) to show that the result holds for the interpolation scales, $\tilde{A}$ and $\tilde{B}$ associated to the polygons $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ respectively. The reiteration theorem shows that these scales coincide with the scales A and B over Int $\tilde{\Pi}_1$ and Int $\tilde{\Pi}_2$ resp. Note that the modified polygons $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ can be choosen arbitrarily close to the original ones Π_1, resp. Π_2; so we can make $\tilde{\Pi}_1 \to \Pi_1$ and $\tilde{\Pi}_2 \to \Pi_2$ (for Hausdorff distance) to obtain the desired result.

If the polygons Π_1 and Π_2 do not satisfy the hypothesis of Proposition 3.3, we can find ourselves in the following unpleasant situation.

EXAMPLE 3.4. Let $\Pi_1 = \{(0, 0), (1, 0), (0, 1)\}$ be the simplex and consider the triple $\bar{A} = \{\ell_1, \ell_1(2^{-m}), \ell_1(2^{-n})\}$. Let A be the interpolation scale associated to this triple and the affine function $\varphi = 1$ (constant function). Choose now the polygon $\Pi_2 = \{(\frac{1}{4}, \frac{1}{2}), (\frac{3}{4}, \frac{1}{2}), (1, 1)\}$ and the triple $\bar{B} = \{\ell_1(2^{-(1/4)m-(1/2)n}), \ell_\infty(2^{-(3/4)m-(1/2)n}), \ell_1(2^{-m-n})\}$ (see Fig. 5). Let B be the interpolation scale associated to Π_2, $\bar{B}$ and the affine function $\varphi = 1$. Both scales overlap. However, due to its bad position, we can not obtain

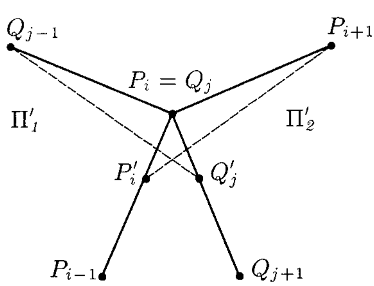

FIG. 4. Points of the 3rd kind.

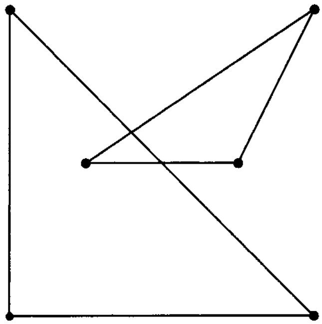

FIGURE 5

the space $\ell_\infty(2^{-34m-12n})$ by interpolation from the tuple $\bar{C} = \{\ell_1, \ell_1(2^{-m}), \ell_1(2^{-n}), \ell_1(2^{-m-n})\}$ and the polygon $\Pi = \{(0,0), (1,0), (0,1), (1,1)\}$ with the affine function $\varphi = 1$.

Remark 3.5. If the ℓ_∞ space were $\ell_\infty(2^{-(5/4)m-(1/2)n})$ placed at the point $(\frac{5}{4}, \frac{1}{2})$ everything works out fine.

In those cases in which polygons Π_1 and Π_2 do not satisfy the hypothesis of Proposition 3.3 there must exist points in the vertices of Π_1 and Π_2 that are not on the vertices of Π, nor in the intersection $\operatorname{Int}\Pi_1 \cap \operatorname{Int}\Pi_2$. Despite of that, these points lie in Int Π. Further, since they are points of Π_1 or Π_2 we can assure that each one of these points lies in the interior of a triangle with vertices in those of Π_1 which also are vertices of Π, in the intersection $\operatorname{Int}\Pi_1 \cap \operatorname{Int}\Pi_2$, and in the vertices of Π_2 which are vertices of Π.

PROPOSITION 3.6. *Assume that the spaces in the above described vertices can be obtained by interpolation with respect to the corresponding triangles and the affine function φ. Then the interpolation scales can be pasted together.*

Proof. It suffices to show that all the spaces in the vertices of Π_1 and Π_2 that are not vertices of Π can be obtained by interpolation from the tuple $\bar{C}$ and the polygon Π.

Let $V_1, ..., V_s$ be the spaces on the vertices of Π_1 and Π_2 that are not vertices of Π and let $I_1, ..., I_s$ the spaces in the intersection associated to the V's (according to the hypothesis). All these spaces are intermediate with respect to the tuple $\bar{C}$. By hypothesis, the V's can be obtained by triangular interpolation between the C's and the I's. Also the points in the intersection, the $I's$, can be obtained by interpolation between the V's and the C's.

Apply Wolff Theorem 2.5 to show that the V's and the I's can be obtained by interpolation from the C's. That is to say, we can generate all the spaces on all vertices of Π_1 and Π_2 not in Π by interpolation from $\bar{C}$. Now apply reiteration theorem, as we indicated before, to show that C contains scales A and B. ∎

An Extension of Proposition 3.3

Let us call *class of uniqueness* a class $\mathscr{C}$ of quasi-Banach spaces continuously embedded in the same ambient space $\mathscr{U}$, which is stable under classical real interpolation and satisfies moreover Lion's uniqueness property: if $\bar{A}=(A_0, A_1)$ and $\bar{B}=(B_0, B_1)$ are two couples in $\mathscr{C}$ satisfying the equalities $\bar{A}_{\theta_0, q_0}=\bar{B}_{\theta_0, q_0}$ and $\bar{A}_{\theta_1, q_1}=\bar{B}_{\theta_1, q_1}$ for some $0<\theta_0<\theta_1<1$ and $q_0, q_1>0$, then the equality $\bar{A}_{\theta, q}=\bar{B}_{\theta, q}$ holds for all $0<\theta<1$ and $q>0$.

For example the class of all Köthe function spaces over a given measure space $(\Omega, \mathscr{A}, \mu)$ is a class of uniqueness, see [1], Corollary 4. Note that members of a class of uniqueness satisfy a more general Lions type uniqueness property, namely:

LEMMA 3.7. *Let Π be a convex polygon and A, B two interpolation scales defined on Π (we assume J-K equivalence property for A and B relatively to Π).*

Suppose that the spaces of A, resp. B, associated with the vertices of Π belong to a given class of uniqueness $\mathscr{C}$. Then the whole scales A, B are included in $\mathscr{C}$. Moreover if A and B coincide over an open subset U of Π, they coincide on the whole of Int Π.

Proof. (a) We suppose first that Π is a triangle $\overline{P_0, P_1, P_2}$, and that U is a subtriangle $\overline{P_0, Q_1, Q_2}$ (where $Q_1 \in \overline{P_0, P_1}$, $Q_2 \in \overline{P_0, P_2}$).

Recall that for every line segment parallel to one of the sides of Π, say $\overline{QR}$ with $Q=(1-\theta) P_0+\theta P_2$, $R=(1-\theta) P_1+\theta P_2$ for some $(0<\theta<1)$ and every point $P \in \overline{QR}$, say $P=(1-\rho) Q+\rho R$ $(0<\rho<1)$, see Fig. 6, we have

$$\bar{A}_{P, q(P)}=\bar{X}_{\rho, q(P)}$$

where $\bar{X}=(X_0, X_1)$, $X_0=(A_0, A_2)_{\theta, q(P)}$, $X_1=(A_1, A_2)_{\theta, q(P)}$ (see [18]). Hence $\bar{A}_{P, q(P)}$ belongs to the class $\mathscr{C}$ (and the interpolation scale A is included in $\mathscr{C}$).

Similarly $\bar{B}_{P, q}=\bar{Y}_{\rho, q(P)}$ with $\bar{Y}=(Y_0, Y_1)$, $Y_0=(B_0, B_2)_{\theta, q(P)}$, $Y_1=(B_1, B_2)_{\theta, q(P)}$ and B is included in $\mathscr{C}$.

Since A and B coincide on U we see that for every $0<\theta<\theta_0$ and $0<\rho<\rho_0(\theta)$ we have $\bar{X}_{\rho, q}=\bar{Y}_{\rho, q}$ (a priori for some $q=q(\theta, \rho)$ but in fact for every q by reiteration). By uniqueness property we deduce $\bar{X}_{\rho, q}=\bar{Y}_{\rho, q}$

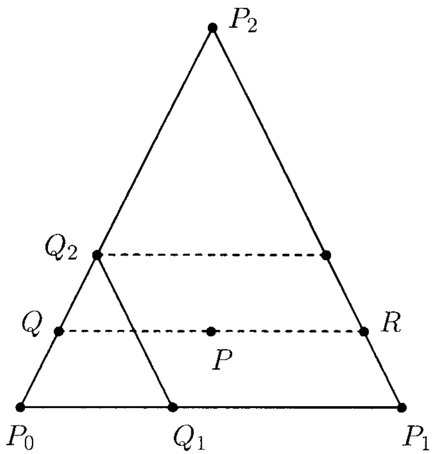

FIGURE 6

for all $0 < \rho < 1$, i.e., $\bar{A}_{P,q} = \bar{B}_{P,q}$ for every $P \in (\mathrm{Int}\, \Pi) \cap B$ where B is the band parallel to the edge $\overline{P_0, P_1}$ generated by U.

Reasoning now with parallels to the edge $\overline{P_0, P_2}$, we see that $\bar{A}_{P,q} = \bar{B}_{P,q}$ for every $P \in \mathrm{Int}\, \Pi$.

(b) We consider now the case of a general polygon Π and a non empty subset U. Let O be a distinguished point of U, see Fig. 7. We may suppose that O belongs to the interior of some triangle $\mathscr{T}$ with vertices in the set of vertices of Π. The space $\bar{A}_{O,q(O)}$ assignated to the point O by the interpolation scale A, is also obtained by real Sparr interpolation from the spaces at the vertices of $\mathscr{T}$, hence belongs to $\mathscr{C}$ by the preceding. Similarly for $\bar{B}_{O,q(O)}$.

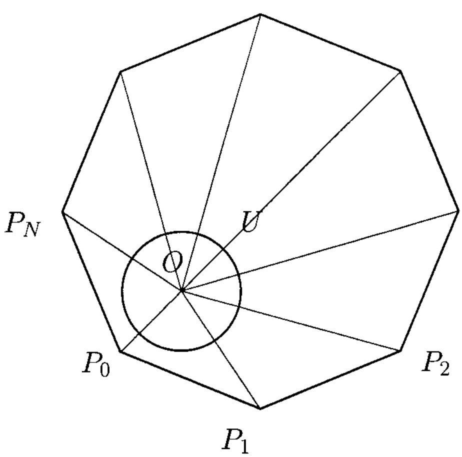

FIGURE 7

We partition the polygon Π into triangles $\mathcal{T}_0 = \overline{O, P_0, P_1}$, $\mathcal{T}_1 = \overline{O, P_1, P_2}$, ..., $\mathcal{T}_N = \overline{O, P_N, P_0}$. Let $\bar{X}^{(k)} = (\bar{A}_{O, q(O)}, A_k, A_{k+1})$, $\bar{Y}^{(k)} = (\bar{B}_{O, q(O)}, B_k, B_{k+1})$. By the Reiteration Theorem, we have for every $P \in \operatorname{Int} \mathcal{T}_k$:

$$\bar{A}^{\Pi}_{P, q(P)} = \bar{X}^{(k)\mathcal{T}_k}_{P, q(P)} \quad \text{and} \quad \bar{B}^{\Pi}_{P, q(P)} = \bar{Y}^{(k)\mathcal{T}_k}_{P, q(P)}.$$

By the preceding we have $\bar{X}^{(k)\mathcal{T}_k}_{P, q(P)} = \bar{Y}^{(k)\mathcal{T}_k}_{P, q(P)} \in C$. Hence the interpolation scales coincide on $\bigcup_k \operatorname{Int}(\mathcal{T}_k)$, with values in $\mathscr{C}$.

If P belongs to one of the segments $\overline{O, P_k}$, $k = 0, ..., N$ it can be represented as barycenter of three points of $\bigcup_k \operatorname{Int}\ (\mathcal{T}_k)$; using the Reiteration Theorem once more, we have the equality

$$\bar{A}^{\Pi}_{P, q(P)} = \bar{B}^{\Pi}_{P, q(P)} \in \mathscr{C}. \quad \blacksquare$$

PROPOSITION 3.8. *Let Π_1, Π_2 be two polygons with $(\operatorname{Int} \Pi_1) \cap (\operatorname{Int} \Pi_2) \neq \varnothing$ and let $\Pi = \operatorname{conv}(\Pi_1 \cup \Pi_2)$.*

Let A, B be two overlapping interpolation scales associated respectively with Π_1 and Π_2, with values in a given class of uniqueness $\mathscr{C}$. We suppose that J-K equivalence property holds for each of the tuples associated by A and B with the vertices of Π_1, Π_2, Π.

Then the interpolation scale C (associated with Π) pastes together the scales A and B.

Proof. Let $\mathscr{P}$ be the closure $\overline{\operatorname{Int} \Pi_1 \cap \operatorname{Int} \Pi_2}$; $(P_i)_{i \in I_1}$ and $(P_i)_{i \in I_2}$ the vertices of Π_1, resp. Π_2, which are also vertices of Π. Note that $(P_i)_{i \in I_1} \cup (P_i)_{i \in I_2}$ is exactly the set of vertices of Π.

Let $\Pi'_1 = \operatorname{conv}\{(P_i)_{i \in I_1} \cup \mathscr{P}\}$ and $\Pi'_2 = \operatorname{conv}\{(P_i)_{i \in I_2} \cup \mathscr{P}\}$. Then $\operatorname{Int} \Pi'_1 \cap \operatorname{Int} \Pi'_2 = \operatorname{Int} \mathscr{P} = \operatorname{Int} \Pi_1 \cap \operatorname{Int} \Pi_2$ and $\operatorname{conv}\{\Pi'_1 \cup \Pi'_2\} = \Pi$. Let A', resp. B' be the interpolation scale associated with Π'_1, resp. Π'_2, whose values at vertices of Π'_1, resp. Π'_2, coincide with those of A, resp. B. By the Reiteration Theorem, A' coincides with A, resp. B' with B, at every point of $\operatorname{Int} \Pi'_1$, resp. $\operatorname{Int} \Pi'_2$.

Note that the polygons Π'_1 and Π'_2 satisfy the hypotheses of Proposition 3.3. (If S is a vertex of Π'_1 which is not a vertex of Π, then $S \neq P_i$, $i \in I_1$; hence S is a vertex of $\mathscr{P}$, and consequently belongs to $\Pi'_1 \cap \Pi'_2$). Hence the interpolation scale C coincides with A' (that is with A) over $\operatorname{Int} \Pi'_1$ and with B' (hence with B) over $\operatorname{Int} \Pi'_2$. By Lemma 3.7, C coincides with A, resp. with B, over the whole of $\operatorname{Int} \Pi_1$, resp. $\operatorname{Int} \Pi_2$; i.e., C pastes together A and B. $\blacksquare$

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